E-Appendix:
Bell's Theorem

Let's recapitulate Bell's thought experiment as described in the text.

In a tropical aquarium we notice (1) that every fish comes in one of two colors, either red or blue and (2) that every fish is also either large or small and (3) that each fish has spots or no spots on it. So each fish has three, binary attributes: (red or blue), (large or small), (spots or no spots). The opposite of a red fish for our purposes is a "not red fish," and is therefore it is a "blue fish," and vice versa. Likewise the "opposite of a big fish" is "not big," a small fish; and the opposite of a spotted fish is a "not spotted," or an unspotted fish.

A Simple Logic Theorem:

> "The number of red fish that are small, plus the number of big fish that are spotted is always greater than or equal to the number of red fish that are spotted. "

This is a special case of a more general statement for ensembles of objects each having three binary attributes, A, B, and C:

$$N(A, \text{not } B) + N(B, \text{not } C) \geq N(A, \text{not } C).$$

We give the proof of this theorem below.

We'll adapt this, following John Bell, in a "physically reasonable way" to our quantum mechanical experiment (a lot of "philosophy" is hidden in the phrase "physically reasonable way"). We again consider the radioactive decay of some particle (source) that produces two outgoing identical particles, of opposite momenta and spins. The particles go out along the x-axis. We assume (though it's not necessary) that the outgoing particles are either spin–up or spin-down (relative to a "z-axis," which could be done by placing the polarizers on the source). We'll call the out-going particles **1** and **2**, and detect them in special detectors separated a large distance apart that we likewise label as 1 and 2. We'll assume that the total spin of our outgoing particles adds to zero which makes an entangled state. By the conservation of angular momentum if particle **1** is detected to have spin "up" then particle **2** must be spin "down," and so vice versa. Therefore, detecting spin down at **2** is equivalent to detecting a spin up particle at **1**. If the outgoing states were actually "not entangled" as they are in classical physics they would take the form of independent definite or pure spin states, either

|**1**, up; **2**, down> , or |**1**, down; **2**, up> .

However, the conservation of angular momentum of the decaying parent particle forces the total spin angular momentum of the state to be zero (we say a "spin eigenstate of spin zero"). This requires that the state be entangled and takes the form:

$$\frac{1}{\sqrt{2}} \left(\; |\mathbf{1}, \text{up}; \mathbf{2}, \text{down}\rangle \quad - \quad |\mathbf{1}, \text{down}; \mathbf{2}, \text{up}\rangle \; \right)$$

(the minus sign here is associated with the net spin angular momentum of the pair of particles which we take to be 0; a plus sign would be angular momentum 1; the pure states are mixtures of total angular momentum 0 and 1).

Each detector can measure the individual spin of the corresponding outgoing particle, relative to the z-axis, which is perpendicular to the x-axis of motion. However, we can slightly rotate either detector and measure the spins relative to any one of three possible angles from the vertical z-axis: "position A" with angle $\theta = 0$ (vertical); "position B" angle, $\theta = \phi$, "position C " angle $\theta = \chi$. So, the list the logical possibilities for the spin measurements we can get for either detector for the three different pre-sets is:

A = spin up, along $\theta = 0$,

Not A = spin down, along $\theta = 0$

B = spin up, relative to the $\theta = \phi$ direction

Not B = spin down, relative to the $\theta = \phi$ direction

C = spin up relative to the $\theta = \chi$ direction

Not C = spin down relative to the $\theta = \chi$ direction.

Note that we define "B" as spin- up in the B position, "not B" as spin-down in the B position (this is part of the "reasonableness" hypothesis, but a loop-hole in the argument: "not spin up" could be interpreted as any state that is not a spin eigenstate of +1/2 and need not be an eigenstate of spin -1/2).

Now here is the key point: we are interested in the number of spin up particles that are produced with the three attributes ("up" relative to $\theta = 0$, or "up" relative to $\theta = \phi$ or up relative to $\theta = \chi$). We define all of these as effectively measured in detector 1. However, in detector 2, if we measure a spin up, it corresponds to spin down in detector 1 by angular momentum conservation, and vice versa. Therefore, measuring, e.g., N(A in 1, not B in 2) logically corresponds to measuring the number of spin-up for $\theta = 0$ in detector 1, and the number of not-spin-up (effectively for detector 1) in detector 2 for $\theta = \phi$, which is the number of number of observed spin-up in detector 2 for $\theta = \phi$ (since spin up in 2 corresponds to spin down in 1).

Summarizing: Our "aquarium" consists of particles with one of three spin directions, with the binary attribute "up" or "down" for each direction. We run the experiment hundreds of thousands of times and count " fish " of different attributes, one in detector 1 and the other in detector 2. Let's now do the experiment!

Let's run the experiment for 100,000 radioactive decays and count the number of spin up particles for detector 1 in the preset A position (angle vertical ($\theta = 0$)) and simultaneously the number spin down particles in detector 2 the preset B position (for angle ($\theta = \phi$)). This is :

$$\text{N(A; not B) = N(spin up } \theta = 0\text{, 1; spin down } \theta = \phi\text{, 2).}$$

Then we repeat the experiment 100,000 times and measure the number of spin up particles for angle $\theta = \phi$ at detector 1, simultaneous with spin downs for detector angle $\theta = \chi$ at detector 2. This is:

$$\text{N(B; not C) = N(spin up } \theta = \phi\text{,1; spin down } \theta = \chi\text{, 2).}$$

Then, we again repeat the experiment 100,000 times measuring the number of spin up particles for detector 1, angle vertical ($\theta = 0$) and simultaneously spin downs for detector angle $\theta = \chi$ at detector 2:

$$\text{N(A; not C) = N(spin up } \theta = 0\text{, 1, spin down } \theta = \chi \text{ at 2).}$$

The simple logical hypothesis is then:

$$\text{N(A; not B) + N(B; not C)} \geq \text{N(A; not C)}$$

which corresponds to:

$$\text{N(spin up } \theta = 0\text{, 1; spin down } \theta = \phi\text{, 2)} + \text{N(spin up } \theta = \phi\text{,1; spin down } \theta = \chi\text{, 2)}$$
$$\geq \text{N(spin up } \theta = 0\text{, 1, spin down } \theta = \chi \text{ at 2)}$$

The question is:

Does quantum mechanics respect this simple and reasonable logical theorem?

Suppose that the states were not entangled. This is similar to the classical case of our friend mailing to us and to our distant (Rigel 3) friend a blue or a red ball; upon opening our package and finding the ball is red we know instantly that the ball on Rigel 3 is blue. But we haven't transmitted any signals faster than the speed of light.

Indeed, if we work out the results for unentangled states, that is, the particles are produced in the states of the form |1, up; 2, down> , or |2, up; 1, down> , each having a 50% probability, then we find that for small angles $\theta = \phi$ and $\theta = \chi$ the numbers of recorded events are proportional to:

N(A, not B) = N(spin up $\theta = 0$, 1; spin down $\theta = \phi$, 2) $\approx \phi^2$

N(B, not C) = N(spin up $\theta = \phi$,1; spin down $\theta = \chi$, 2) $\approx \phi^2 + \chi^2$

N(A, not C) = N(spin up $\theta = 0$, 1, spin down $\theta = \chi$, 2) $\approx \chi^2$

Note: to make these formulas as simple as possible, the angles measured in Detector 1 are defined relative to the up z-axis; Detector 2 is rotated by π and we measure its angles relative to the down z-axis. *All angles will be measured in radians.*

Checking our inequality, we see that $\phi^2 + \phi^2 + \chi^2 \geq \chi^2$. Indeed, this is a valid inequality because ϕ^2 is always positive. Un-entangled states, the kind that would apply to classical physics, work just fine and agree with the inequality we expect for tropical fish.

But what happens in the real world of quantum theory? Here the state is entangled, a mixture of the two possibilities up-down plus down-up. We now find (calculation below):

N(spin up $\theta = 0$, 1; spin down $\theta = \phi$, 2) $\approx \phi^2$

N(spin up $\theta = \phi$,1; spin up $\theta = \chi$, 2) $\approx (\phi - \chi)^2$

N(spin up $\theta = 0$, 1, spin down $\theta = \chi$, 2) $\approx \chi^2$

And we now check the inequality again: $\phi^2 + (\phi - \chi)^2 \geq \chi^2$. But doing a little algebra, we see that this requires $\phi^2 - \phi\chi \geq 0$. *The latter result is not true in general.* All we need to do is choose both angles small, positive and $\chi \geq \phi$. (remember, angles at detector 2 are measured relative to a rotation of π of the detector… the rotation just makes it easier to look at the math, but the physics is the same) . The Bell inequality is violated by quantum physics!

The entanglement is indeed the source of the violation of the inequality. It leads to the expression N(spin up $\theta = \phi$,1; spin up $\theta = \chi$, 2) $\approx$ $(\phi - \chi)^2$ [entangled] versus N(spin up $\theta = \phi$,1; spin up $\theta = \chi$, 2) $\approx$ $\phi^2 + \chi^2$ [unentangled] which leads to the violation of the inequality.

Proof of the Logic Theorem:

The essential Bell Inequality is the general statement we can make about an ensemble of many objects that each can have three attributes, A, B, and C:

$$N(A, \text{not } B) + N(B, \text{not } C) \geq N(A, \text{not } C)$$

Here's the proof.

Proof: The proof involves considering three conditions at once, such as N(A, not B,C), "the number of red fish that are small (not big) and that are spotted." Since the numbers of fish in any such category is always ≥ 0, (e.g., both N(A, not B, C) ≥ 0 N(not A, B, C) ≥ 0) then their sum is:

$$N(A, \text{not } B, C) + N(\text{not } A, B, \text{not } C) \geq 0$$

And if we don't specify an attribute, e.g., B, then it can be either B or not B:

$$N(A, \text{not } C) = N(A, \text{not } B, \text{not } C) + N(A, B, \text{not } C)$$

Add the equality to both sides of the inequality :

$$N(A, \text{not } B, C) + N(\text{not } A, B, \text{not } C) + N(A, \text{not } B, \text{not } C) + N(A, B, \text{not } C) \geq N(A, \text{not } C)$$

and use: N(A, not B) = N(A, not B, C)+ N(A, not B, not C), and N(B, not C) = N(A, B, not C) + N(not A, B, not C), so that we have proved:

$$N(A, \text{not } B) + N(B, \text{not } C) \geq N(A, \text{not } C)$$

This is a result using standard logic.

Derivation of the Quantum Result:

We will now see how to adapt the logic theorem in a “physically reasonable way” to a quantum mechanical situation. Consider the radioactive decay of some particle (source) that produces two outgoing identical particles, 1 and 2, of opposite momenta and spins. We will assume that our source has a polarizing filter so that the outgoing particles are either spin –up or spin down. If one of the particles is detected and found to be spin up then the other, by the conservation of angular momentum must be spin down, and so forth.

The quantum state is entangled and we’ll assume it takes the form:

$$| S > \; = \frac{1}{\sqrt{2}} \; (|1, up; 2, down> \;\; - \;\; |1, up; 2, down>)$$

(the minus sign here is associated with the total spin angular momentum of the pair of electrons which we assume to be 0, though it could be chosen as 1 with the same results).

We place detectors on either sides of the radioactive source. The detectors measure the spins of the outgoing particles relative to the z axis. We can measure any one of three possible spin angles, up with angle $\theta = 0$, or angle, $\theta = \phi$, or a different angle $\theta = \chi$

Let us define the logical possibilities:

A = spin up detector 1,
Not A = spin down detector 1 = not spin up detector 1,
B = spin up relative to the $\theta = \phi$ direction detector 1
Not B = spin down relative to the $\theta = \phi$ direction detector 1,
C = spin up relative to the $\theta = \chi$ direction detector 1,

Not C = spin down relative to the $\theta = \chi$ direction detector 1,

The main point to keep in mind is that spin “up” in detector 1 is equivalent to spin “down” in detector 2. This means that “not spin up” in detector 1 is equivalent to spin “up” in detector 2. We measure simultaneously the different cases for one of the particles in the 1 detector and the 2 detector. So, our “aquarium” consists of particles with one of three spin up directions, and we sample them on the left and on the right. The logical theorem is then:

$$\text{N}(\mathbf{1}, \text{spin up } \theta = 0\,;\, \mathbf{2}, \text{spin up } \theta = \phi) + \text{N}(\mathbf{1}, \text{spin up } \theta = \phi\,;\, \mathbf{2}, \text{spin up } \theta = \chi)$$
$$\geq \text{N}(\mathbf{1}, \text{spin up } \theta = 0\,;\, \mathbf{2}, \text{spin up } \theta = \chi)$$

We will calculate the quantum mechanical result below. What we obtain is as follows

$$\text{N}(\mathbf{1}, \text{spin up } \theta = 0\,;\, \mathbf{2}, \text{spin up } \theta = \phi) = \frac{1}{2}\sin^2\frac{\phi}{2} \approx \frac{\phi^2}{8}$$

$$\text{N}(\mathbf{1}, \text{spin up } \theta = \phi\,;\, \mathbf{2}, \text{spin up } \theta = \chi) = \frac{1}{2}\sin^2\left(\frac{\phi}{2} - \frac{\chi}{2}\right) \approx \frac{(\phi-\chi)^2}{8}$$

$$\text{N}(\mathbf{1}, \text{spin up } \theta = 0\,;\, \mathbf{2}, \text{spin up } \theta = \chi) = \frac{1}{2}\sin^2\frac{\chi}{2} \approx \frac{\chi^2}{8}$$

We compute these results from the theory of spinors.

First consider $\text{N}(\mathbf{1}, \text{spin up } \theta = 0\,;\, \mathbf{2}, \text{spin up } \theta = \phi)$ measurement. The detector 1 is measuring a particle in the state |up>. Detector 2 is measuring an up state that is rotated through the angle $\theta = \phi$ about the x axis. Hence the state that is being detected is:

$$|\,\text{S'}> = |1, \text{up}> \left(\left(\cos(\phi/2)\,|\,2, up > + i\sin(\phi/2)\,|\,2, down >\right)\right)$$

The quantum amplitude to observe this state is just $< S' \,|\, S >$:

$$< S' | S > = \frac{1}{\sqrt{2}} < 1, up | 1, up > \left(i \sin(\phi/2) < 2, down | 2, down > \right) = \frac{i}{\sqrt{2}} \left(\sin(\phi/2) \right)$$

The probability is the square: $|< S' | S >|^2 = \frac{1}{2}\left(\sin(\phi/2)\right)^2 \approx \frac{\phi^2}{8}$, for small angle.

Let's now consider the case N(**1**, spin up $\theta = \phi$; **2**, spin up $\theta = \chi$) . The state being detected is |1, up> that has been rotated through $\theta = \phi$ and an |2,up> , rotated through $\theta = \chi$, so:

$$| S'> = \left(\cos(\phi/2) | 1, up > + i \sin(\phi/2) | 1, down >\right)\left(\left(\cos(\chi/2) | 2, up > + i \sin(\chi/2) | 2, down >\right)\right.$$

The quantum amplitude to observe this state is

< S | S' >

$$= \frac{i}{\sqrt{2}} \cos(\phi/2) < 1, up | 1, up > \left(\sin(\chi/2) < 2, down | 2, down >\right)$$

$$- \frac{i}{\sqrt{2}} \sin(\phi/2) < 1, down | 1, down > \left(\cos(\chi/2) < 2, up | 2, up >\right)$$

$$= \frac{i}{\sqrt{2}} \sin((\phi - \chi)/2)$$

The probability is the square: $|< S' | S >|^2 = \left(\frac{1}{\sqrt{2}} \sin((\phi - \chi)/2)\right)^2 \approx \frac{(\phi - \chi)^2}{8}$.

The N(**1**, spin up $\theta = 0$; **2,** spin up $\theta = \chi$) is simply N(**1,** spin up $\theta = 0$; **2**, spin up $\theta = \phi$) with $\theta = \phi$ replaced by $\theta = \chi$:

N(**1,** spin up $\theta = 0$; **2**, spin up $\theta = \chi$) $= \frac{1}{2}\left(\sin/\chi/2)\right)^2 \approx \frac{\chi^2}{8}$

Where we again have noted the approximate result for small angles, measured in radians. Also, if we did the experiment with photons all angles in the expressions should be multiplied by 2 and the sign of chi flipped.

Now, let's test Bell's Inequality:

$$\frac{\phi^2}{8} + \frac{(\phi - \chi)^2}{8} \geq \frac{\chi^2}{8}$$

Doing a little algebra, this requires: $\frac{\phi^2}{4} - \frac{\phi\chi}{4} \geq 0$.

But this inequality is false (simply take $1 \geq \chi \geq \phi \geq 0$).

We have shown that the entangled quantum state violates the simple idea of classical logic for a system of objects with three attributes.

The effect comes entirely from entanglement. Suppose that the states were always disentangled, either |1, up; 2, down> or |2, up; 1, down> . Then we would find:

N(spin up $\theta = 0$, 1; spin down $\theta = \phi$, 2) $= \frac{1}{2}\sin^2\frac{\phi}{2} \approx \frac{\phi^2}{8}$

N(spin up $\theta = \phi$,1; spin up $\theta = \chi$, 2) $= \frac{1}{2}\sin^2\left(\frac{\phi}{2}\right) + \frac{1}{2}\sin^2\left(\frac{\chi}{2}\right) \approx \frac{\phi^2 + \chi^2}{8}$

N(spin up $\theta = 0$, 1, spin down $\theta = \chi$, 2) $= \frac{1}{2}\sin^2\frac{\chi}{2} \approx \frac{\chi^2}{8}$

And the inequality is always preserved:

$$\frac{\phi^2}{8} + \frac{\phi^2 + \chi^2}{8} \geq \frac{\chi^2}{8}$$

Notes:

Our present explanation of Bell's theorem here is adapted from the delightful website: http://www.upscale.utoronto.ca/PVB/Harrison/BellsTheorem/BellsTheorem.html